2026-02-24

Fermionic Gaussian States (Olivier)

Dave : Digital signatures with classical shadows

Last episode: • Majorana Fermions

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

• strings

space of bitstrings

$$\mathcal{M}_{2n} = \{ \mu(\vec{v}) \mid \vec{v} \in \mathbb{F}_{2n} \}$$

bitstrings

• group structures → Lie algebra

$$[\mu(\vec{v}), \mu(\vec{v}')]$$

Hamming weight

• with closed subalgebra for $\|\vec{v}\| \leq 2$

• Superselection rule based on parity $p(\vec{v}) = \vec{v}^T\vec{v}$

• Nature does not contain odd parity observables; Hamiltonians commute with the parity operator

Today, define the even subgroups

$$\mathcal{M}_{2n}^{+} = \{ \mu(\vec{v}) \mid p(\vec{v}) = 0 \}$$

Majorana Cliffords

$$C_{2n} = \{ \Gamma \in SU(2^{2N}) \mid \Gamma \mu(\vec{v}) \Gamma^{+} = c(\vec{v}) \mu(S\vec{v}) \}$$

phase matrix

(both depends on Γ)

with : $p(S\vec{v}) = p(\vec{v})$

but valid for any μ

Q: unique A: not sure, may be

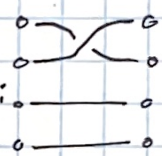
So the matrix S is parity preserving ~~so~~ can verify S is orthogonal

$$(S\vec{v})^T(Sv) = v^T S^T S v = v^T v \rightarrow \boxed{S^T S = 1}$$

Gottesmann-Knill theorem for parity-preserving Majorana Clifford:

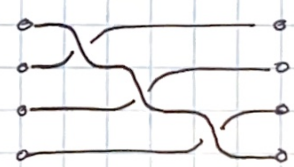
$$\boxed{C_{2n}^{+} \text{ is generated by braiding operator}}$$

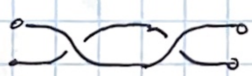
<u>Braids</u>

$\mathbb{1}$:  σ_1:  $\sigma_1^{\dagger}$: 

Basically a permutation group where the order of permutation matters

σ_2:  σ_3:  $\sigma_4 = \sigma_3 \sigma_2 \sigma_1$

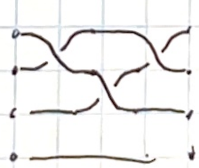
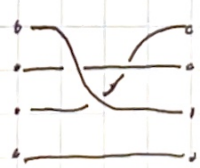
$=$ 

$\sigma_1^{\dagger}\sigma_1 = 1 =$ 

Note that: commute if they don't intersect

$$\boxed{\sigma_3 \sigma_1 = \sigma_1 \sigma_3}$$

Also:

$\sigma_1 \sigma_2 \sigma_1 =$  $=$  $\rightarrow \boxed{\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}}$

Definition

$$B_n = \left\{ \sigma_i \;\middle|\; \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & 1 \leq i \leq n-2 \end{array} \right\}$$

Back to Majorana,
you can find the following (not shown):

$$\sigma_k = (1 + \gamma_{k+1} \gamma_k)/\sqrt{2}$$

$$\sigma_k^\dagger = (1 - \gamma_{k+1} \gamma_k)/\sqrt{2}$$

Apply on Majoranas

$$\sigma_0 = (1 + \gamma_1 \gamma_0)/\sqrt{2}$$

$$\begin{aligned} \sigma_0 \gamma_0 \sigma_0^\dagger &= (1 + \gamma_1 \gamma_0) \gamma_0 (1 - \gamma_1 \gamma_0) \\ &= (\gamma_0 + \gamma_1)(1 - \gamma_1 \gamma_0) \\ &= \cancel{\gamma_0} - \underbrace{\gamma_0 \gamma_1}_{2} \gamma_0 + \gamma_1 - \cancel{\gamma_1 \gamma_1 \gamma_0} \\ &= \frac{2\gamma_1}{2} = \gamma_1 \end{aligned}$$

$$\sigma_0 \gamma_1 \sigma_0^\dagger = -\gamma_0$$

The intuition is that the sign is $\mathcal{X}$ vs $\mathcal{X}$

Back to classical simulability

Fermionic Gaussian States

three equivalent definition

① $$\rho = \frac{e^{-\hat{H}}}{Z} \qquad Z = Tr[e^{-\hat{H}}]$$

where $\hat{H}$ is a free-fermion hamiltonian $H = \frac{i}{4} \gamma^T K \gamma$

$$= \frac{i}{4} \sum_{ij} \gamma_i K_{ij} \gamma_j$$

② ρ can be generated by free fermionic evolution of the Fock vacuum

$$e^{-iH} |0\rangle\langle 0| e^{iH} \qquad \text{where } H$$

related to
Naimark's dilation theorem: ~~Partial~~
any mixed state can be $Tr_A(|\psi\rangle\langle\psi|_{AB})$
~~depend on the~~ dimension ~~is~~
depends

③ ρ is entirely characterized by correlation matrix

$$\Lambda_{ab} = \frac{i}{2} Tr(\rho[\gamma_a, \gamma_b]) = i\langle \gamma_a, \gamma_b \rangle - i\delta_{ab}$$

Classical simulability will be acheived through keeping track of the correlations. But before we show that time evolution is a rotation

$$U = e^{iH} \qquad U^\dagger \gamma_a U = R\vec{\gamma} = \sum_b R_a{}^b \gamma_b$$

definition of A

$$\boxed{\begin{aligned} \frac{d}{dt}\gamma_j(t) &= i[H, \gamma_j(t)] \\ &= \sum_k A_{jk} \gamma_k \end{aligned}}$$

real antisym

$$\vec{\gamma}(t) = \underbrace{e^{At}}_{R} \vec{\gamma}(0) \qquad R \in SO(2N)$$

$$\Lambda = \frac{i}{4}\,\mathrm{Tr}\left(|0\rangle\langle 0| \, [\gamma_a, \gamma_b] \right)$$

$$\Lambda'_{ab} = \frac{i}{4}\,\mathrm{Tr}\left(|0\rangle\langle 0| \, U^\dagger [\gamma_a, \gamma_b] U \right)$$

$$= \frac{i}{4}\,\mathrm{Tr}\left(|0\rangle\langle 0| \, [U^\dagger \gamma_a U, \, U^\dagger \gamma_b U] \right)$$

$$= \frac{i}{4}\,\mathrm{Tr}\left(|0\rangle\langle 0| \left[\sum_i R_a^{\,c} \gamma_c , \sum_d R_b^{\,d} \gamma_d \right] \right)$$

$$= \frac{i}{4} \sum_i \sum_d R_a^{\,c} R_b^{\,d} \, \mathrm{Tr}\left[|0\rangle\langle 0| \, [\gamma_c, \gamma_d] \right]$$

$$= \frac{i}{4} \, R \Lambda R^T \qquad R \in SO(2N)$$

time evolution → only rotate the correlation matrix

Next time: Wick's theorem