Simon Verret 03/2026

QARS - Impurity solver

Plan

- Fermionic Gaussian States
- Wick's theorem
- ~~Stabilizers (the parallel with)~~
- ~~Impurity models~~ ~~Ressource theory~~
- Impurity models

Fermionic Gaussian states

Dirac Fermions

(1) $$|\vec{x}\rangle = (a_1^\dagger)^{x_1}(a_2^\dagger)^{x_2}\cdots(a_3^\dagger)^{x_n}\,|\text{vacuum}\rangle \qquad \vec{x}\in\{0,1\}^n = \mathbb{F}_n$$

where

(2) $$\{a_i, a_j\} = a_i a_j + a_j a_i = 0$$

(3) $$\{a_i^\dagger, a_j^\dagger\} = a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger = 0$$

(4) $$\{a_i^\dagger, a_j\} = a_i^\dagger a_j + a_j a_i^\dagger = \delta_{ij}$$

Majorana fermions

(5) $$c_{2j-1} = a_j + a_j^\dagger$$

(6) $$c_{2j} = -i(a_j - a_j^\dagger)$$

(7) $$\{c_i, c_j\} = c_i c_j + c_j c_i = 2\delta_{ij} \quad \& \quad c_i^2 = 1$$

As Pauli strings:

little-endian / big-endian

(8) $$c_{2j-1} = X\,\underbrace{Z\,Z\cdots Z}_{j-1} \quad\leftrightarrow\quad \underbrace{Z\,Z\cdots Z}_{j-1}\,X\,\underbrace{I\cdots I}_{n-j}$$

(9) $$c_{2j} = Y\,\underbrace{Z\,Z\cdots Z}_{j-1} \quad\quad \underbrace{Z\,Z\cdots Z}_{j-1}\,Y\,\underbrace{I\cdots I}_{n-j}$$

<u>Gaussian unitaries</u> map Majoranas to majoranas

(10) $$U c_j U^\dagger = \sum_{k=1}^{2n} R_{jk} c_k \qquad RR^T = \mathbb{I}$$ (AKA Bogoliubov transformation related to Hartree Fock)

form a group $\mathcal{C}_n$ which coincide with $O(2n)$ (modulo phase) generated by operators

(11) $$U = \exp\left(\frac{\theta}{2} c_p c_q\right) \qquad \text{and} \qquad c_p$$ (flip sign for a $c_q \neq c_p$)

acting like

(12) $$U c_p U^\dagger = \cos\theta\, c_p - \sin\theta\, c_q$$

(13) $$U c_q U^\dagger = \sin\theta\, c_p + \cos\theta\, c_q$$

<u>Gaussian state</u> is obtained from vaccuum by ~~an~~ an gaussian unitary

(14) $$|\varphi\rangle = U|0^n\rangle \qquad \text{with } U \in \mathcal{C}_n$$

specified up to a phase by the covariance matrix

(15) $$M_{p,q} \equiv \cancel{\xi} -\frac{i}{2}\langle\varphi|(c_p c_q - c_q c_p)|\varphi\rangle = \cancel{\langle c_p c_q\rangle} \Rightarrow$$ real, anti-symmetric

example: the Fock state $|\vec{y}\rangle$ has block diagonal M:

(16) $$M_{\vec{y}} = \bigoplus_{j=1}^{n} \begin{pmatrix} 0 & (-1)^{y_j} \\ (-1)^{y_j+1} & 0 \end{pmatrix}$$

note: $(\cancel{\xi}_p c_q)^\dagger = c_q^\dagger c_p^\dagger = c_q c_p = -c_p c_q$

pair is anti-hermitian
adding i makes it real
anti symmetric

ex:

(17) $$|000\rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad |011\rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(18) $$\cancel{\xi} = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & 0 & 0 & 1 & & \\ & & 1 & 0 & 0 & \\ & & & 0 & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix} \qquad = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & 0 & & & \\ & 0 & 0 & -1 & & \\ & & +1 & 0 & 0 & \\ & & & 0 & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix}$$

Gaussian state

(19) $$|\varphi\rangle = U|\vec{y}\rangle$$

has covariance matrix

(20) $$M_\varphi = R M_{\vec{y}} R^T$$ where $R \in O(2)$

defined by

(21) $$U c_p U^\dagger = \sum_{q=1}^{2n} R_{pq} c_q$$

Valid covariance matrices statisfy

(22) $$M^2 = -I$$

Any real antisymmetric satisfying this encodes some gaussian fermionic state.

Expectation values

consider observables ~~$c(\vec{x})$~~ in the basis of Majorana monomials (Majorana strings)

(23) $$c(\vec{x}) = c_1^{x_1} c_2^{x_2} \dots c_{2n}^{x_n} \qquad \vec{x} \in \{0,1\}^{2n} = \mathbb{F}_{2n}$$

expectation value is computable from covariance matrix only (Wick's theorem)

(24) $$\langle\varphi| c(\vec{x}) |\varphi\rangle = \mathrm{Pf}\big(i \underbrace{M_\varphi[\vec{x}]}\big)$$

Pfaffian $\in \mathbb{C}$ (to be defined)

submatrix where ~~x~~ we use only rows and column j if $x_j \neq 0$

(25) ex: $\langle c_1 c_2 c_3 c_4\rangle = \langle c_1 c_2\rangle\langle c_3 c_4\rangle - \langle c_1 c_3\rangle\langle c_2 c_4\rangle - \langle c_1 c_4\rangle\langle c_2 c_3\rangle$

Pfaffian is analogous to the determinant, ~~in fact~~ but for $2k \times 2k$ skew-symmetric matrices,

(26) $$A = \underbrace{\begin{pmatrix} 0 & a_{12} & a_{13} & \cdots \\ -a_{12} & 0 & a_{23} & \cdots \\ -a_{13} & -a_{23} & 0 & \ddots \\ \vdots & \vdots & & \ddots \end{pmatrix}}_{2n} \Bigg\} 2k = n$$

The pfaffian is

(27) $$Pf(A) = \frac{1}{2^k k!} \sum_{\sigma \in S_n} (-1)^\sigma a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2k-1)\sigma(2k)}$$

ex:

$\dim = 2 \rightarrow Pf(A) = a_{12}$

$\dim = 3 \rightarrow Pf(A) = 0$

$\dim = 4 \rightarrow Pf(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$

$$\begin{aligned} \dim = 6 \rightarrow Pf(A) = & \; a_{12}a_{34}a_{56} - a_{12}a_{35}a_{46} + a_{12}a_{36}a_{45} \\ & - a_{13}a_{24}a_{56} + a_{13}a_{25}a_{46} - a_{13}a_{26}a_{45} \\ & + a_{14}a_{23}a_{56} - a_{14}a_{25}a_{36} + a_{14}a_{26}a_{35} \\ & - a_{15}a_{23}a_{46} + a_{15}a_{24}a_{36} - a_{15}a_{26}a_{34} \\ & + a_{16}a_{23}a_{45} - a_{16}a_{24}a_{35} + a_{16}a_{25}a_{34} \end{aligned}$$

we see a recursive pattern

(28) $$Pf(A) = \sum_{j=2}^{2n} (-1)^j a_{1j} \, Pf(A_{\hat{1}\hat{j}})$$

← matrix with row and columns 1 and j removed

note that $pf(A)^2 = \det(A)$.

Note also that when the matrix is the covariance of some ~~free ferm~~ gaussian state terms ~~simplify~~,

(29) $$\cancel{M_{pq}M_{rs} = \frac{1}{4}\left(\langle c_p c_q\rangle - \langle c_q c_p\rangle\right)\left(\langle c_r c_s\rangle - \langle c_s c_r\rangle\right)}$$

$$\cancel{= \frac{1}{4}\left(\langle c_p c_q\rangle\langle c_r c_s\rangle - \langle c_p c_q\rangle\langle c_s c_r\rangle\right)}$$

(30) $$M_{pq} = -\frac{i}{2}\langle\varphi|(c_p c_q - c_q c_p)|\varphi\rangle = -\frac{i}{2}\left(\langle\varphi|c_p c_q|\varphi\rangle - \delta_{pq}\right)$$

Wick's theorem (or Isserlis theorem)

Consider a normal distribution (gaussian)

(1) $$p(\vec{x}) = \underbrace{\frac{1}{Z}}_{\text{normalization constant}} \exp\Big(-\frac{1}{2}\vec{x}^T \underbrace{\Sigma^{-1}}_{\text{covariance matrix}} \vec{x}\Big) \qquad \text{with } \Sigma_{ij} \equiv \langle x_i x_j \rangle$$

Now consider

(2) $$Z[\vec{b}] = \int d^n\vec{x} \, \exp\Big(-\frac{1}{2}\vec{x}^T\Sigma^{-1}\vec{x} + \vec{b}^T\vec{x}\Big)$$

note that $Z[0]$ is the normalization constant Z. ~~and~~ We call $Z[\vec{b}]$ the "generating functional" because derivatives generate moments, as below. The term "functional" is abusive here, but makes sense when the integral is a path integral in field theory. $\vec{b}$ is a "source field"

(3) $$\frac{\partial}{\partial b_i} Z[\vec{b}] = \int d^n x \, \exp\Big(-\frac{1}{2}\vec{x}^T\Sigma^{-1}\vec{x} + \vec{b}^T\vec{x}\Big)\frac{\partial}{\partial b_i}\vec{b}^T\vec{x} \qquad \vec{b}^T\vec{v} = \sum_j b_j v_j$$

(4) $$= \int d^n x \, \exp\Big(-\frac{1}{2}\vec{x}^T\Sigma^{-1}\vec{x} + \vec{b}^T\vec{x}\Big)\, x_i$$

$$\cancel{\frac{\partial}{\partial b_i} Z[\vec{b}]\Big|_{\vec{b}=0} \propto \langle x_i \rangle}$$

(5) $$= Z\int d^n x \, \frac{1}{Z}\exp\Big(-\frac{1}{2}\vec{x}^T\Sigma^{-1}\vec{x} + \vec{b}^T\vec{x}\Big)\, x_i$$

(6) $$\frac{\partial}{\partial b_i} Z[\vec{b}]\Big|_{\vec{b}=0} = Z\int d^n x \, p(\vec{x})\, x_i$$

(7) $$= Z\langle x_i \rangle$$

Similarly for any powers of x_i

(8) $$\frac{\partial^m Z[\vec{b}]}{\underbrace{\partial b_i \partial b_j \partial b_k \cdots}_{m}}\Bigg|_{\vec{b}=0} = Z\langle \underbrace{x_i x_j x_k \cdots}_{m} \rangle$$

Let us now show that this can be written as

(9) $$\frac{\partial^m Z[\vec{b}]}{\partial b_i \partial b_j \cdots \partial b_k}\Bigg|_{\vec{b}=0} = Z\frac{\partial^m}{\partial b_i \partial b_j \cdots \partial b_k}\exp\Big(-\frac{1}{2}\vec{b}^T\Sigma\vec{b}\Big)\Bigg|_{\vec{b}=0}$$

the key is a change of variable under the integral, which constitute one of a few important Gaussian integral of field theory

(10) $$Z[\vec{b}] = \int d^n x \exp\left(-\frac{1}{2}\vec{x}^T\Sigma^{-1}\vec{x} + \vec{b}^T\vec{x}\right) \quad \vec{x} \to \vec{x} + \Sigma\vec{b}$$

(11) $$= \int d^n x \exp\left(-\frac{1}{2}(\vec{x}+\Sigma\vec{b})^T\Sigma^{-1}(\vec{x}+\Sigma\vec{b}) + \vec{b}^T\vec{x}\right)$$

(12) $$= \int d^n x \exp\left(-\frac{1}{2}\vec{x}^T\Sigma^{-1}\vec{x} \cancel{-\frac{1}{2}\vec{b}^T\vec{x}} \cancel{-\frac{1}{2}\vec{x}^T\vec{b}} - \frac{1}{2}\vec{b}^T\Sigma\vec{b} \cancel{+\vec{b}^T\vec{x}}\right)$$

(13) $$= Z[0]\exp\left(-\frac{1}{2}\vec{b}^T\Sigma\vec{b}\right)$$

As a result, we have (9), an with (8), it implies

$$\cancel{\langle x_i x_j \dots x_k\rangle = \frac{1}{Z}\frac{\partial^m}{\partial b_i \partial b_j \dots \partial b_k}}$$

(14) $$\langle x_i x_j \dots x_k\rangle = \frac{\partial^m}{\partial b_i \partial b_j \dots \partial b_k}\exp\left(-\frac{1}{2}\vec{b}^T\Sigma\vec{b}\right)$$

(15) $$= \frac{\partial^m}{\partial b_i \partial b_j \dots \partial b_k}\left(1 - \frac{1}{2}\vec{b}^T\Sigma\vec{b} + \frac{1}{2!\,2}(\vec{b}^T\Sigma\vec{b})^2 + \dots\right)\Bigg|_{\vec{b}=0}$$

The derivative only keep terms containing all relevant b_i and evaluation at $\vec{b}=0$ remove all terms containing extra b_i's. The remaining terms can only contain even powers of b_i and a pattern emerges:

$$\underbrace{\langle x_i x_j \dots x_k\rangle}_{m} = \sum_{p\in\text{pairings}} \prod_{P} \Sigma_P \qquad \Sigma_{ij} \equiv \langle x_i x_j\rangle$$

$$= \Sigma_{ij}\Sigma_{kl}\Sigma_{mn}\dots + \Sigma_{ik}\Sigma_{jl}\dots + \dots$$

which is Wick's theorem! (Actually this is Isserlis theorem) The nuance with fermions is that since the fermionic field (which would take the place of the x_i here) anti commute, the sign changes accordingly in the product of $\Sigma_{ij}\Sigma_{kl}$. Proving this follow the exact steps used above but requires to introduce grassmman variables (anti-commuting scalars) for the source fields $\vec{b}$.

QARS - Impurity model Simon Verret 03/2026

Impurity model: free fermion bath coupled to a small, strongly interacting impurity

$$H = H_0 + H_{imp.}$$

m fermionic modes with interaction

$n-m$ fermionic modes free

n modes total

History:

- Kondo '64 problem: resistivity upturn at low T (1933)
- Solved by Anderson '61, Wilson '75: localization, NRG numerical renormalization group (predecessor of TN)
- Used in DMFT to treat correlations in materials impurity model with bath fixed self-consistently with ~~that a~~ lattice of such impurities
- Bauer 2015: could us a quantum computer to solve larger impurities in DMFT

Why care? Impurity models sit at the frontier between ~~exactly solvable~~ efficiently simulable free fermions systems and QMA-complete systems

Hubbard 2D: QMA-complete

Hubbard 1D: exact Bethe ansatz

Hubbard ∞D: exact DMFT

Our best approximation to Hubbard 2D is DMFT and CDMFT (cluster impurity)

Today: Gosset & Bravyi 2017: complexity of Quantum impurity problems

Impurity models in the Majorana basis:

(1) $$H_0 = \frac{i}{4} \sum_{ij=1}^{2n} h_{ij} c_i c_j$$ energy scale s.t. $\|h\| \leq 1$

(2) $$H_{imp} = \sum_{\substack{\vec{x} \in \{0,1\}^m \\ |\vec{x}| \text{ even}}} g_{\vec{x}} \, c_1^{x_1} c_2^{x_2} \cdots c_m^{x_m}$$ where $c_i c_j + c_j c_i = 2\delta_{ij}$, $c_i^2 = 1$

The bath diagonalized by (Gaussian unitary)

(3) $$H_0 = \sum_{j=1}^{n} \epsilon_j b_j^\dagger b_j + \underbrace{E_0}_{\text{ignore this}}$$ where $b_j^2 = 0$, $b_i b_j^\dagger + b_j^\dagger b_i = \delta_{ij}$

with $b_j = U a_j U^\dagger$

the spectrum is $\{\epsilon_j\}$ where $0 \leq \epsilon_j \leq 1$ because of the scale we chose. we identify

(4) $$\epsilon_0 = 0$$

(5) $$\omega = \text{gap} \; \cancel{= \min \epsilon_j \text{ s.t. } \epsilon_j \neq 0} \quad \epsilon_j \geq \omega > 0 \;\; \forall j = 1, 2, \ldots$$

Define the covariance matrix

(6) $$C_{pq} = \langle \psi | b_p^\dagger b_q | \psi \rangle$$

where $|\psi\rangle$ is the ground state of the impurity then Gosset & Bravyi's main result is:

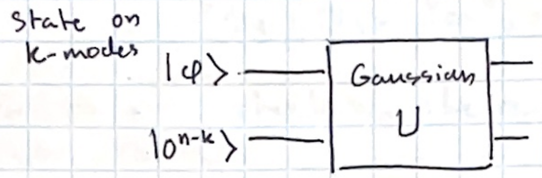
Theorem: eigenvalues σ_i of C decay exponentially

(7) $$\sigma_j \leq \text{const.} \exp\left(- \frac{j}{\cancel{14} \, m \log(2\omega^{-1})} \right) \qquad \sigma_1 \geq \sigma_2 \geq \ldots$$

7 in the paper, 14 in the talk

with no dependence on n.

Implies that the ground state has a concise representation, namely



δ-approximation to the ground state
= superposition of 2^k gaussian states

the 2K basis vectors on the left map to 2^k gaussian states : the <u>gaussian rank</u>, then because of the theorem, we can use

$$k = \mathcal{O}(1) \cdot m \log(2\omega^{-1}) \cdot \left[\log \delta^{-1} + \log m + \log\log 2\omega^{-1} \right]$$

independant of n. The proof is roughly that for each vanishing eigenvalue of C the eigenstate

$$C \vec{v} \approx 0$$

~~corre~~ there correspond a Fermi mode

$$B = \sum_{j=1}^{n} v_j b_j$$

such that $\langle \psi | B^{\dagger} B | \psi \rangle = 0$. ~~With that concise representation comes~~ So a Gaussian unitary $U^{\dagger}$ can represent the ground state in a bassic where most B are unoccupied.

With that concise representation, we can devise an algorithm:

<u>Theorem</u>: there is a classical algorithm to compute the ground energy of a quantum impurity model within tolerance ϵ with runtime

$$\text{complexity} \sim n^3 \exp\left(\mathcal{O}(m \log^3 (m \epsilon^{-1}))\right)$$

the resulting ground state is a superposition of

$$\chi = \exp\left[\mathcal{O}(m \log^3(m\epsilon^{-1}))\right]$$

gaussian states. Here is a sketch of the algorithm:

1. diagonalize bath $\mathcal{O}(n^2) \rightarrow H_0 = \sum_j \epsilon_j b_j^\dagger b_j$

2. discretize $H_0 \rightarrow \tilde{H}_0 = \sum_{j=1}^{n} E_j b_j^\dagger b_j$ with $\epsilon_j \leq E_j \leq \epsilon_j + \Delta$
this modifies the hamiltonian and the GS up to error $\mathcal{O}(\epsilon)$ for spacing

$$\Delta = \frac{\epsilon}{m \log^2(m\epsilon^{-1})}$$

$\|H_0 - \tilde{H}\|$ can be large but we only care about the few excitations subspace

3. This equally spaced grid $E_j - E_i = \Delta$ introduce degenerancy in the bath, ~~in which case~~ at most $1/\Delta = m\epsilon^{-1}\log^2(m\epsilon^{-1})$. When the bath have such degenerancy we can decouple all except m/Δ modes $= m^2 \epsilon^{-1} \log(m\epsilon^{-1})$

   moreover, the decoupling unitary commutes with number operator $b_j^\dagger b_j$ so we can further truncate $\leq m \log^2(m\epsilon^{-1})$

Computing the smallest eigenvalue in this subspace yield the desired ground state

Time evolution is harder: Brod & Childs show it can't be simulated unless BPP = BQP

Appendix

From fermionic hamiltonians to Majorana

(1) $$H = \sum_{ij} (t_{ij}\, c^\dagger_i c_j + \Delta_{ij}\, c_i c_j + h.c.)$$

(2) $$= \sum_{ij} (t_{ij}\, c^\dagger_i c_j + \Delta_{ij}\, c_i c_j + \Delta^*_{ij}\, c^\dagger_j c_i + t^*_{ij}\, c^\dagger_j c_i)$$

(3) $$= \sum_{ij} (t_{ij}\, c^\dagger_i c_j + \Delta_{ij}\, c_i c_j + \Delta^*_{ij}\, c_i c^\dagger_j + t^*_{ij}\, (\delta_{ij} - c_i c^\dagger_j))$$

(4) $$= \sum_{ij} \begin{pmatrix} c^\dagger_i & c_i \end{pmatrix} \begin{pmatrix} t_{ij} & -\Delta^*_{ij} \\ \Delta_{ij} & -t^*_{ij} \end{pmatrix} \begin{pmatrix} c_j \\ c^\dagger_j \end{pmatrix} + \sum_i t_{ii}$$

note: not exactly Namby because spin is not there

Fermions de Majorana

(5) $$\gamma_{2j-1} = \gamma^o_j = c_j + c^\dagger_j$$

(6) $$\gamma_{2j} = \gamma^e_j = -i(c_j - c^\dagger_j)$$

$$\Rightarrow \begin{pmatrix} \gamma^o \\ \gamma^e \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}}_{M} \begin{pmatrix} c_j \\ c^\dagger_j \end{pmatrix}$$

inverse transform:

(7) $$\underbrace{\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}}_{M^\dagger} \underbrace{\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}}_{M} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} c_j \\ c^\dagger_j \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \gamma^o \\ \gamma^e \end{pmatrix}$$

(8) $$\begin{pmatrix} c^\dagger_i & c_i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \gamma^o & \gamma^e \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

Hamiltonian

(9) $$H = \frac{1}{4} \sum_{ij} \begin{pmatrix} \gamma^o_i & \gamma^e_i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} t_{ij} & -\Delta^*_{ij} \\ \Delta_{ij} & -t^*_{ij} \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \gamma^o_j \\ \gamma^e_j \end{pmatrix} + \sum_i t_{ii}$$

(10) $$= \frac{1}{4} \sum_{ij} \begin{pmatrix} \gamma^o_i & \gamma^e_i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} t_{ij} - \Delta^*_{ij} & i t_{ij} + i\Delta^*_{ij} \\ \Delta_{ij} - t^*_{ij} & i\Delta_{ij} + i t^*_{ij} \end{pmatrix} \begin{pmatrix} \gamma^o_j \\ \gamma^e_j \end{pmatrix} + \sum_i t_{ii}$$

(11)
$$= \frac{1}{4}\sum_{ij} \begin{pmatrix} \gamma_i^o & \gamma_i^e \end{pmatrix} \begin{pmatrix} (t_{ij} - t_{ij}^*) + (\Delta_{ij} - \Delta_{ij}^*) & i(t_{ij} + t_{ij}^*) + i(\Delta_{ij} + \Delta_{ij}^*) \\ -i(t_{ij} + t_{ij}^*) + i(\Delta_{ij} + \Delta_{ij}^*) & (t_{ij} - t_{ij}^*) - (\Delta_{ij} - \Delta_{ij}^*) \end{pmatrix} \begin{pmatrix} \gamma_j^o \\ \gamma_j^e \end{pmatrix} + \sum_i t_{ii}$$

$$t = t' + i t''$$
$$\Delta = \Delta' + i\Delta''$$

(12)
$$= \frac{1}{4}\sum_{ij} \begin{pmatrix} \gamma_i^o & \gamma_i^e \end{pmatrix} \begin{pmatrix} 2i(t''_{ij} + \Delta''_{ij}) & 2i(\Delta'_{ij} + t'_{ij}) \\ 2i(\Delta'_{ij} - t'_{ij}) & 2i(t''_{ij} - \Delta''_{ij}) \end{pmatrix} \begin{pmatrix} \gamma_j^o \\ \gamma_j^e \end{pmatrix} + \sum_i t_{ii}$$

(13)
$$= \frac{i}{\cancel{4}\,2}\sum_{ij} \begin{pmatrix} \gamma_i^o & \gamma_i^e \end{pmatrix} \begin{pmatrix} \mathrm{Im}(t+\Delta)_{ij} & \mathrm{Re}(t+\Delta)_{ij} \\ -\mathrm{Re}(t-\Delta)_{ij} & \mathrm{Im}(t-\Delta)_{ij} \end{pmatrix} \begin{pmatrix} \gamma_j^o \\ \gamma_j^e \end{pmatrix} + \sum_i t_{ii}$$

$$= \frac{i}{2}\sum_{pq} \gamma_p h_{pq} \gamma_q$$

example: Hubbard

$$H_U = U\sum_i a^\dagger_{i\uparrow} a_{i\uparrow} a^\dagger_{i\downarrow} a_{i\downarrow} = U\sum_i n_{i\uparrow} n_{i\downarrow}$$

Majorana

$$c_{2j-1} = a_j + a_j^\dagger \qquad \rightarrow \qquad a_j = \frac{1}{2}(c_{2j-1} + i c_{2j})$$

$$c_{2j} = -i(a_j - a_j^\dagger) \qquad a_j^\dagger = \frac{1}{2}(c_{2j-1} - i c_{2j})$$

done

$$H_U = \frac{U}{8}\sum_i \underbrace{(c_{2j-1} - i c_{2j})(c_{2j-1} + i c_{2j})}(c_{2j-1} - i c_{2j})(c_{2j-1} + i c_{2j})$$

$$n_{i\uparrow} = c_{2j-1}c_{2j-1} - i c_{2j}c_{2j-1} + i c_{2j-1}c_{2j} + c_{2j}c_{2j}$$

$$= 2(1 + i\, c_{2j-1}c_{2j})$$

$$= \frac{4U}{8}\sum_i (1 + i\, c_{2j-1\uparrow}c_{2j\uparrow})(1 + i\, c_{2j-1\downarrow}c_{2j\downarrow})$$

$$= \frac{U}{2}\sum_i \left(1 + i(c_{2j-1\uparrow}c_{2j\uparrow} + c_{2j-1\downarrow}c_{2j\downarrow}) - c_{2j-1\uparrow}c_{2j\uparrow}c_{2j-1\downarrow}c_{2j\downarrow}\right)$$

$$H_U = \underbrace{\frac{U}{2}}_{\text{constant}} + \underbrace{\frac{U}{2}\sum_{i\sigma} c_{2j-1\sigma}c_{2j\sigma}}_{\text{chemical potential term}} + \frac{U}{2}\sum_{i\sigma} c_{2j-1\uparrow}c_{2j-1\downarrow}c_{2j\uparrow}c_{2j\downarrow}$$

example: Hubbard

$$H_U = U \sum_i a^\dagger_{i\uparrow} a_{i\uparrow} a^\dagger_{i\downarrow} a_{i\downarrow} \quad = \quad U \sum_i n_{i\uparrow} n_{i\downarrow}$$

Majorana

$$c_{2j-1} = a_j + a_j^\dagger \qquad \rightarrow \qquad a_j = \frac{1}{2}(c_{2j-1} + i c_{2j})$$

$$c_{2j} = -i(a_j - a_j^\dagger) \qquad\qquad a_j^\dagger = \frac{1}{2}(c_{2j-1} - i c_{2j})$$

done

$$H_U = \frac{U}{8} \sum_i \underbrace{(c_{2j-1} - i c_{2j})(c_{2j-1} + i c_{2j})}(c_{\downarrow 2j-1} - i c_{2j})(c_{2j-1} + i c_{2j})$$

$$n_{i} = c_{2j-1} c_{2j-1} - i c_{2j} c_{2j-1} + i c_{2j-1} c_{2j} + c_{2j} c_{2j}$$

$$= 2(1 + i c_{2j-1} c_{2j})$$

$$= \frac{4U}{8} \sum_i (1 + i c_{2j-1\uparrow} c_{2j\uparrow})(1 + i c_{2j-1\downarrow} c_{2j\downarrow})$$

$$= \frac{U}{2} \sum_i (1 + i(c_{2j-1\uparrow} c_{2j\uparrow} + c_{2j-1\downarrow} c_{2j\downarrow}) - c_{2j-1\uparrow} c_{2j\uparrow} c_{2j-1\downarrow} c_{2j\downarrow})$$

$$H_U = \underbrace{\frac{U}{2}}_{\text{constant}} + \underbrace{\frac{U}{2} \sum_{i\sigma} c_{2j-1\sigma} c_{2j\sigma}}_{\text{chemical potential term}} + \frac{U}{2} \sum_{i\sigma} c_{2j-1\uparrow} c_{2j-1\downarrow} c_{2j\uparrow} c_{2j\downarrow}$$