{
41

QARS Feb 10 2026, Majorana operators and FGS.

Table of contents

1 Majorana operators

Start from ~~[illegible]~~ ~~[illegible]~~ fermionic operators. Consider N fermionic modes, def $2N$ operators $c_i, c_i^\dagger$ s.t.

$$\{c_i, c_i^\dagger\} = \delta_{ij} \qquad \{c_i^\dagger, c_j^\dagger\} = 0 \qquad \{c_i, c_j\} = 0$$

define vacuum $|0\rangle$ s.t. $c_i|0\rangle = 0 \ \forall i$

Def Majorana operators
$$\gamma_{2j} = c_j + c_j^\dagger$$
$$\gamma_{2j+1} = i(c_j - c_j^\dagger)$$

Properties: $\gamma_j^\dagger = \gamma_j \ \forall j \qquad \gamma_j^2 = \mathbb{I} \ \forall_j$

$$\cancel{c_j^2} + c_j c_j^\dagger + c_j^\dagger c_j + \cancel{c_j^{\dagger 2}} \quad \mathbb{I}$$

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

Majorana strings and Group Structure

Define a Majorana strings (up to a sign)

$$\mu(\vec{v}) = (i)^{v^T \omega_L v} \gamma_0^{v_0} \gamma_1^{v_1} \dots \gamma_{2n}^{v_{2n}}$$

$2n$ ⊕ of finite field of 2 elements

$$v \in \mathbb{F}_2^{2n}$$

ω_L is the simplectic matrix $\begin{pmatrix} 0 & 0 & & 0 \\ 1 & 0 & & 0 \\ 1 & 1 & \ddots & 0 \\ 1 & & \cdots 1 & 0 \end{pmatrix}$

$\langle u, v \rangle = u^T \omega v$ defines a simplectic product (more on this later)

Majorana Group:

$$\mathcal{M}_{2n} = \left\{ (i)^a \mu(\vec{v}) \;\middle|\; a \in \mathbb{Z}_4, \vec{v} \in \mathbb{F}_2^{2n} \right\}$$

Composition rules

addition in $\mathbb{F}_2^{2n}$ (mod 2)

$$\mu(\vec{v})\mu(\vec{v}') = \xi(v, v')\, \mu(v + v')$$

$$\xi(v, v') = (-1)^{v^T \omega_L v' + f(v, v')} (i)^{v^T \omega v'}$$

$$f(v, v') = (v^T \omega_L v)(v'^T \omega_L v') + v^T \omega v' (v^T \omega_L v + v'^T \omega_L v' + 1)$$

$$\omega = \begin{pmatrix} 0 & 1 & & 1 \\ 1 & 0 & 1 & \\ & 1 & 0 & 1 \\ 1 & & 1 & 0 \end{pmatrix} = \vec{1} - \mathbb{I} \qquad \xi(v, v) = 1$$

Commutation relations:

$$\mu(v)\mu(v') = (-1)^{v^T \omega v'} \mu(v')\mu(v)$$

$\langle v, v' \rangle$ defines commutation relations.

Aside: as a Lie algebra

Add the Lie bracket $[\mu(v), \mu(v')]$

$= \mu(v)\mu(v') - \mu(v')\mu(v)$

Notice that $[\gamma_i, \gamma_j] = \gamma_i\gamma_j - \gamma_j\gamma_i = \begin{cases} 0 & i=j \\ 2\gamma_i\gamma_j & i \neq j \end{cases}$

linears generate bilinears (quadratic terms)

Now $[\gamma_i, \gamma_j\gamma_k] = \gamma_i\gamma_j\gamma_k - \gamma_j\gamma_k\gamma_i$

if
$i \neq j \neq k \Rightarrow 0$ — linear-bilinear generate linear terms
$i=j$: $2\gamma_k$
$i=k$: $-2\gamma_j$

$[\gamma_i\gamma_j, \gamma_k\gamma_l] = ijkl - klij$

$= 2\delta_{il}\, jk + 2\delta_{jk}\, il - 2\delta_{ik}\, jl - 2\delta_{jl}\, ik$

size: $\binom{2N}{2}$

bilinears are closed under the Lie bracket, subalgebra of the full Majorana algebra. Actually, it is so(2n)

1 last example: $[\gamma_i\gamma_j\gamma_k\gamma_l, \gamma_m\gamma_n\gamma_o\gamma_p]$ say $l=m$

$\gamma_i\gamma_j\gamma_k\gamma_n\gamma_o\gamma_p - \gamma_l\gamma_n\gamma_o\gamma_p\gamma_i\gamma_j\gamma_k\gamma_l = \gamma_i\gamma_j\gamma_k\gamma_n\gamma_o\gamma_p$

degree 6 term.

No longer closed, can generate all strings. in general: A is p-linear, B is q-linear, r in common

$AB = (-1)^{pq-r} BA$

Parity and Super selection

def weight $w(\mu(\vec{v}))$: Hamming weight of $\vec{v}$

parity $P(\mu(\vec{v}))$: parity of $w(\mu)$

equivalently: $p(\vec{v}) = v^T v = v^T \cdot \vec{1}$

as an operator/generator: $\mu(\vec{1}) = (-1)^{\hat{n}}$

$= 0$ $\quad 0$

operators which commute with $\mu(\vec{1})$ preserve parity

Parity superselection

What is a superselection?

selection rule: Two states $|\psi_1\rangle, |\psi_2\rangle$ are separated by a selection rule if $\langle\psi_1| H |\psi_2\rangle = 0$ for some Hamiltonian H

and a superselection rule if $\langle\psi_1| A |\psi_2\rangle = 0$ for all physical observables A.

Cannot distinguish btw $|\psi_1\rangle + |\psi_2\rangle$ and $|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$

fermions: $(-1)^{\hat{n}}$ imposes a superselection rules btw states of different parities.

remember: $\{c_i, c_j\} = 0$
$\{c_i^\dagger, c_j^\dagger\} = 0$
$\{c_i, c_j^\dagger\} = 0$

why? thought experiment.

Alice and Bob share a state

$$|\psi\rangle = (1 + c_B^\dagger)|0\rangle$$

~~Bob~~ Alice wants to send a bit to ~~Bob~~ Bob. If

0 : she applies $\mathbb{I}$

1 : She applies $i(c_A^\dagger - c_A)$

$$|\psi_0\rangle = (1 + c_B^\dagger)|0\rangle \qquad |\psi_1\rangle = i(c_A^\dagger - c_A)(1 + c_B^\dagger)|0\rangle$$
$$= (1 - c_B^\dagger)\, c_A^\dagger |0\rangle$$

Bob measures $O = (1 + c_B + c_B^\dagger)$

$$O|\psi_0\rangle = (1 + c_B^\dagger + c_B)(1 + c_B^\dagger)|0\rangle = 2|0\rangle + 2c_B^\dagger|0\rangle$$
$$= 2|\psi_0\rangle$$

$$O|\psi_1\rangle = (1 + c_B + c_B^\dagger)((1 - c_B^\dagger)\, c_A^\dagger|0\rangle$$
$$= 0$$

Both eigenstates of $\hat{O}$ w/ diff expvals.

loss of causality

Now remember: $\mu(v)\mu(v') = (-1)^{v^T w v'}\mu(v')\mu(v)$.

Only strings which commute with $\mu(\hat{1})$ are physical (preserve parity) $\Rightarrow$ $v^T w \hat{1}$ is even.

$$w\hat{1} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}\begin{pmatrix} \vdots \end{pmatrix} = \begin{pmatrix} \vdots \end{pmatrix}$$ since basis is even-dimensional

So $v^T w \hat{1} = v^T \hat{1} = p(v)$ $\Leftarrow$ only even strings commute w/ parity operator!

Evenness is preserved under composition

def

$$\mathbb{F}_2^{2n,+} := \{ v \in \mathbb{F}_2^{2n} \mid p(v) = 0 \}$$

$$\mathcal{M}_{2n}^{+} = \{ (i)^a \mu(v) \mid a \in \mathbb{Z}_4, v \in \mathbb{F}_2^{2n,+} \}$$

is a group.

$\mathcal{M}_{2n}^{-}$ is not a group.

aside: evenness is preserved under the Lie bracket, so it is also a subalgebra.

Majorana Clifford Group

$$\mathcal{C}_{2n} = \{ \Gamma \text{ unitary} \mid \Gamma \mu(v) \Gamma^{\dagger} = c(v) \mu(S(v)) \}$$

$$c: \mathbb{F}_2^{2n} \longrightarrow \pm 1$$

$$S: \mathbb{F}_2^{2n} \rightarrow \mathbb{F}_2^{2n}$$

i.e. maps majoranas to majoranas up to a phase.

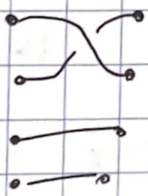
parity-preserving Clifford: $\mathcal{C}_{2n}^{p}$

$$p(\vec{v}) = p(S\vec{v}) = (S\vec{v})^T (Sv)$$

$$= v^T S^T S v \Rightarrow S \text{ is orthogonal}$$

Gottesman-Knill Theorem for p-Cliffords:
Any element of $\mathcal{C}_{2n}^{p}$ can be generated by products of braiding operators.

Braiding group: (B_n)

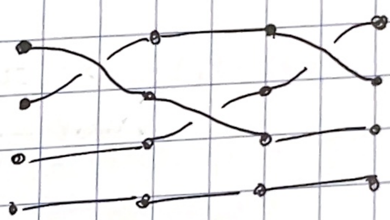
$\mathbb{I} =$  $\sigma_1 =$  $\sigma_2 =$  $\sigma_3 =$ 

composition

$\sigma_1 \sigma_2 \sigma_3 =$ 

properties:

$\sigma_1 \sigma_3 =$  $= \sigma_3 \sigma_1$

$\sigma_1 \sigma_2 \sigma_1 =$  $=$  $= \sigma_2 \sigma_1 \sigma_2$

and $\sigma_2 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 \sigma_3$

B_n is entirely specified by these relations

$$B_n = \left\{ \sigma_1, \dots \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & 1 \le i \le n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| \geq 2 \end{array} \right\}$$

Majorana Cliffords (again)

def:

$$\tau_k = (1 + \gamma_{k+1}\gamma_k)/\sqrt{2} \qquad \tau_k^\dagger = (1 - \gamma_{k+1}\gamma_k)$$

$$\tau_k \tau_j = \tau_j \tau_k \text{ if } |k-j| \geq 2$$

can verify that $\tau_k \tau_{k+1} \tau_k = \tau_{k+1} \tau_k \tau_{k+1}$

thus τ are a representation of the braid group.

Intuition: consider the effect of

$$\tau_k \, \mu(v) \, \tau_k^\dagger \qquad \text{say } k=0$$

τ generates C^P_{2n}

$$\cancel{(1 + i\gamma_1\gamma_0)\gamma_2(1 - i\gamma_1\gamma_0) = \gamma_2}$$

$$\cancel{(1 + i\gamma_1\gamma_0)\gamma_1(1 - i\gamma_1\gamma_0)} =$$

$$\cancel{(\gamma_1 + i\gamma_1\gamma_0\gamma_1)(1 - i\gamma_1\gamma_0)}$$

$$\cancel{= \gamma_1 - i\gamma_1\gamma_1\gamma_0 \ldots \gamma_1}$$

$$\cancel{(\gamma_1 - i\gamma_0)(1 - i\gamma_1\gamma_0)}$$

$$= \gamma_1 - i\gamma_0 \cancel{- i\gamma_0 - \gamma_0\gamma_1\gamma_0}$$

001 → 001

010 → -100

100 → 010

$$(1 + \gamma_1\gamma_0)\gamma_2(1 - \gamma_1\gamma_0) = \gamma_2$$

$$(1 + \gamma_1\gamma_0)\gamma_1(1 - \gamma_1\gamma_0) =$$

$$(\gamma_1 - \gamma_0)(1 - \gamma_1\gamma_0) = \gamma_1 - \gamma_1 - \gamma_0 - \gamma_0 = -\gamma_0$$

$$(1 + \gamma_1\gamma_0)\gamma_0(1 - \gamma_1\gamma_0)$$

$$= (\gamma_0 + \gamma_1)(1 - \gamma_1\gamma_0) = \gamma_0 + \gamma_1 + \gamma_1 - \gamma_0 = \gamma_1$$

permutation of bitstring with some sign depending on orientation of braid.

Continued on page

Fermionic Gaussian states

ρ is a FGS if

$$\rho = \frac{e^{-\hat{H}}}{Z} \quad \text{where} \quad Z = \mathrm{Tr}[e^{-\hat{H}}]$$

$\hat{H}$ is quadratic in the Majoranas

~~[illegible]~~

equivalently: can be generated from the vacuum by free fermion evolution

def $H = \underbrace{\frac{i}{4}\vec{\gamma}^T K \vec{\gamma}}_{\text{quadratic}}$ $\quad$ K is real antisymmetric

$$\vec{\gamma} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{2n} \end{pmatrix}$$

$U = e^{-iHt}$ acts linearly on Majoranas

$$U^\dagger \gamma_a U = \sum_b R_a^b \gamma_b \neq \text{[redacted]}$$

def the covariance matrix $\Gamma^{(s)}_{ab} = \frac{i}{4}\mathrm{Tr}(\rho[\gamma_a, \gamma_b])$

$$\Gamma^{(0)}_{ab} = \frac{i}{4}\mathrm{Tr}(|0\rangle\langle 0|[\gamma_a, \gamma_b])$$ $2n \times 2n$ matrix

evolve under U:

$$\Gamma^{(t)}_{ab} = \frac{i}{4}\mathrm{Tr}(U|0\rangle\langle 0|U^\dagger[\gamma_a, \gamma_b]) = \frac{i}{4}\mathrm{Tr}(|0\rangle\langle 0|U^\dagger[\gamma_a, \gamma_b]U)$$

$$= \frac{i}{4}\mathrm{Tr}(|0\rangle\langle 0|[U^\dagger\gamma_a U, U^\dagger\gamma_b U]) = \frac{i}{4}\mathrm{Tr}(|0\rangle\langle 0|[\sum_c R_a^c\gamma_c, \sum_d R_b^d \gamma_d])$$

$$= \frac{i}{4}\sum_c\sum_d R_a^c R_b^d \,\mathrm{Tr}(|0\rangle\langle 0|[\gamma_c, \gamma_d]) = \frac{i}{4} R\,\Gamma^{(0)} R^T$$

$R \in SO(2N)$

Date

Continued on page

Date