ZX Calculus

1. Intro (arXiv:2012.13966, see also Wikipedia and Pennylane intro)
    1. A graphical approach that better lends itself to manipulation of quantum linear maps than does quantum circuits
        1. Eg. Goal: Show what circuit of Sec 5.5 represents
            1. Show, don't write
    2. PyZX (Pysics, like Physics) package can calculate these automatically
        1. https://www.youtube.com/watch?v=iC-KVdB8pf0
    3. Approach: Picked several example circuits, will pick up the tools as we go
        1. Paper has more details and rigour. My goal is to give a small demonstration of what ZX Calculus can do, hopefully with some intuition.
2. Basic maps (Sec. 2.1, 2.2, and 3.1; references are to arXiv:2012.13966)
    1. Z gate
    2. S gate
    3. T Gate
    4. X gate
    5. Four basis states
3. Generalization
    1. 2->2
    2. 2->1
    3. Full generalization (Sec. 3.1)
    4. Functor (Sec. 7)
    5. Link to tensor networks (pg. 12 to 13)
    6. Identity (Sec. 4.1)
    7. Fuse (Sec. 4.1)
4. CNOT (Sec. 3.2)
5. Input/Output Swap (Sec. 3.3)
    1. Generalize identity (Sec. 4.1)
6. 5.1 GHZ preparation
    1. Hadamard (Sec. 3.6)
    2. Solve (Sec. 5.1)
7. 5.2 Pauli Pushing
    1. Pi copy (Sec. 4.2)
    2. Solve (Sec. 5.2)

Z Gate

$-\boxed{Z}- = \begin{pmatrix}1&0\\0&-1\end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$

$= R_Z(\pi)$

$= e^{-\frac{i\pi}{2}}(|0\rangle\langle 0| + e^{i\pi}|1\rangle\langle 1|)$

$\mathbb{C}^2 \to \mathbb{C}^2$

drop global scalar $\longrightarrow$ —(π)— → blue

input — output

1 leg ⟶ 1 leg

$-\boxed{S}- = \begin{pmatrix}1&0\\0&i\end{pmatrix} = |0\rangle\langle 0| + i|1\rangle\langle 1| = R_Z(\frac{\pi}{2}) \longrightarrow$ —($\frac{\pi}{2}$)—

$-\boxed{T}- = \begin{pmatrix}1&0\\0&e^{\frac{i\pi}{4}}\end{pmatrix} \longrightarrow$ —($\frac{\pi}{4}$)—

$-\boxed{X}- = \begin{pmatrix}0&1\\1&0\end{pmatrix} = |+\rangle\langle+| - |-\rangle\langle-| = |+\rangle\langle+| + e^{i\pi}|-\rangle\langle-|$

$\longrightarrow$ —(π)— → red

$|+\rangle$? $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ → ~~Gen~~ 1 output, 0 inputs

(0)— = O— → blue

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \longrightarrow$ (π)—

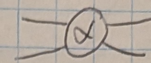
$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \longrightarrow$ (0)— → red

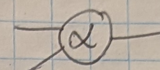
$|1\rangle \longrightarrow$ (π)—

$\langle +|$? $\longrightarrow$ 0 output, 1 input $\xrightarrow{\text{transpose}}$ —O

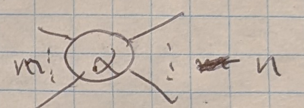
$\langle -| \xrightarrow{\text{dagger}}$ —(−π) = —(π)

Generalize

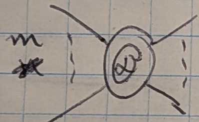
$|00\rangle\langle 00| + e^{i\alpha}|11\rangle\langle 11| \xrightarrow[\text{z basis}]{\text{2 in, 2 out}}$ 

$|0\rangle\langle 00| + e^{i\alpha}|1\rangle\langle 11| \xrightarrow[\text{z basis}]{\text{1 out, 2 in}}$ 

$|\overbrace{0\cdots0}^{n}\rangle\langle\overbrace{0\cdots0}^{m}| + e^{i\alpha}|\overbrace{1\cdots1}^{n}\rangle\langle\overbrace{1\cdots1}^{m}|$

$\hookrightarrow$ 

m legs $\longrightarrow$ ~~m~~ n legs



$= |\overbrace{+\cdots+}^{n}\rangle\langle\overbrace{+\cdots+}^{m}| + e^{i\alpha}|\overbrace{-\cdots-}^{n}\rangle\langle\overbrace{-\cdots-}^{m}|$

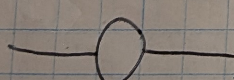
Category

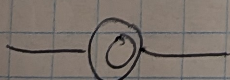
X & Z spiders $D \xrightarrow{\text{functor}} [\![D]\!]$

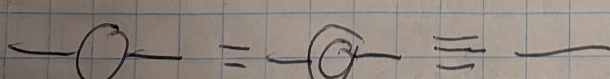
~~matrix~~ linear map (ie. matrix)
$\hookrightarrow$ not necessarily unitary

Tensor Networks

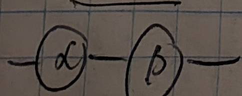
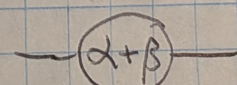
Identity

 $= ?$ $(\rightarrow |0\rangle\langle 0| + |1\rangle\langle 1| = Id)$

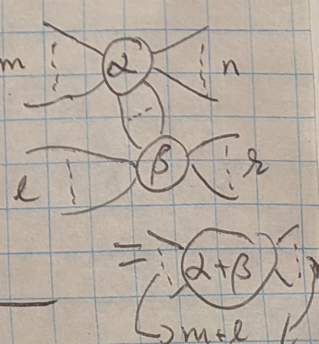
 $= ?$ $(= Id)$



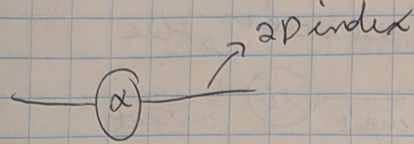
Fuse

 $\longrightarrow R_z(\beta)R_z(\alpha) = R_z(\alpha+\beta) \longrightarrow$ 

generalizes



Tensor Network

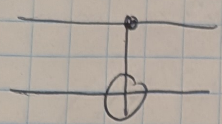
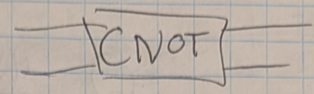


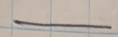
$$\left(\textcircled{\alpha}\right)^{j_1\cdots j_n}_{i_1\cdots i_m} = \begin{cases} 1 & \text{if all } i_1,\ldots,i_m, j_1,\ldots,j_n = 0 \\ e^{i\alpha} & \text{if all} \qquad \text{"} \qquad = 1 \\ 0 & \text{otherwise} \end{cases}$$

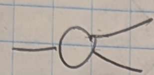
$$\left(\textcircled{\textcircled{\alpha}}\right)^{j_1\cdots j_n}_{i_1\cdots i_m} = \frac{1}{\sqrt{2}} \begin{cases} 1+e^{i\alpha} & \text{if } \overbrace{\bigoplus_k}^{\text{XOR}} i_k \oplus \bigoplus_l j_l = 0 \rightarrow \text{i.e. parity} \\ 1-e^{i\alpha} & \text{if} \qquad \text{"} \qquad = 1 \end{cases}$$

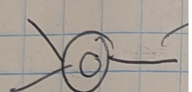
But, this belies their usefulness.

CNOT

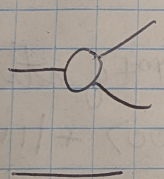
 2 in 2 out $\longrightarrow$ 

Need:  $\longrightarrow \mathbb{1}$

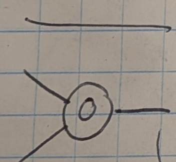
 $\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\left(= |00\rangle\langle 0| + |00\rangle\langle 1| \right)$

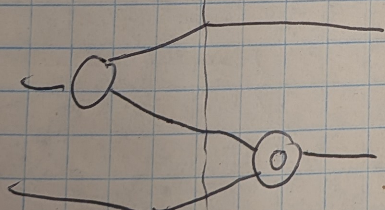
 $\frac{1}{\sqrt{2}} \left(\mathbb{1}\ X \right)$

<u>kronecker</u>

 $\Longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbb{1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$

top bottom

 $\Longrightarrow \mathbb{1} \otimes \frac{1}{\sqrt{2}} \left(\mathbb{1}\ X \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}\ X & 0\ 0 \\ 0\ 0 & \mathbb{1}\ X \end{pmatrix}$



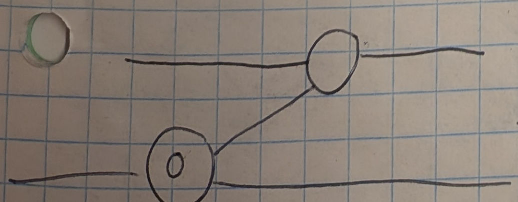
<u>mat mul</u>

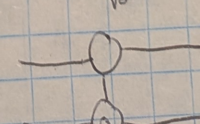
(order that makes sense)

Right to left

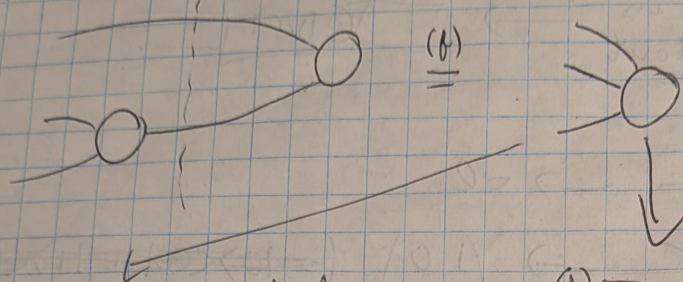
$\frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}\ X & 0\ 0 \\ 0\ 0 & \mathbb{1}\ X \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & X \end{pmatrix}$

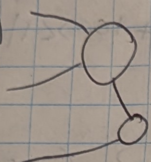
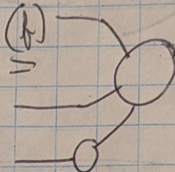
$= \frac{1}{\sqrt{2}}$ CNOT

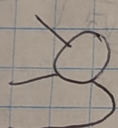
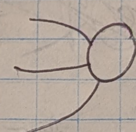
 $\Longrightarrow$?

$\hookrightarrow$ also CNOT $\Longrightarrow$ 

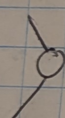
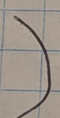
Input/Output Swap



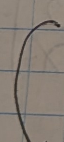
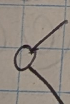
$\overset{(f)}{=}$  freely swap ⟷ input + output $\overset{(f)}{=}$ 

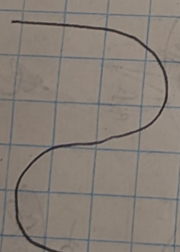
$\overset{(id)}{\equiv}$  freely swap ⟷ $\overset{(id)}{=}$ 

GHZ state: $|000\rangle + |111\rangle$

 $\equiv$  → generalization of identity

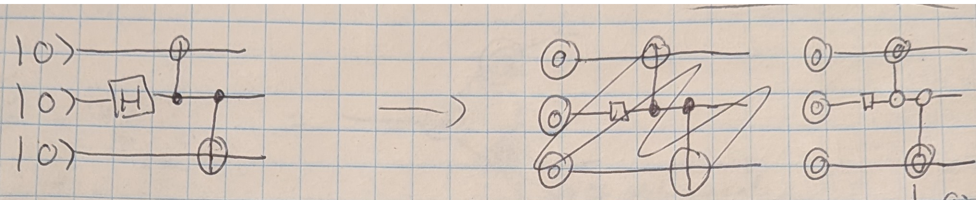
Bell pair: $|00\rangle + |11\rangle$

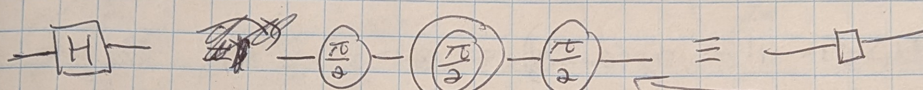
 $\equiv$ 

 $=$ 

→ see notebook after 5.5

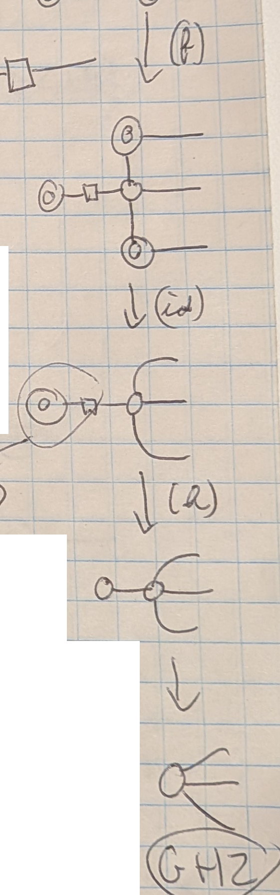
5.1





$\hookrightarrow = \text{phase} \cdot R_z(\frac{\pi}{2})\ R_x(\frac{\pi}{2})\ R_z(\frac{\pi}{2})$

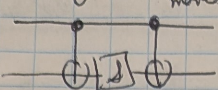
(Euler angles)



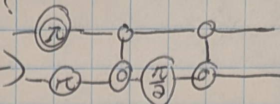
$|+\rangle = H|0\rangle$

5.2

$X \otimes Z$ how it moves through?

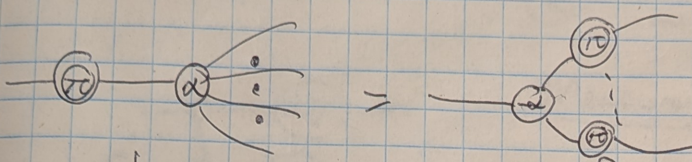


$\longrightarrow$



↳ stabilizer propagation

π − copy



up to global scalar

$(|0\cdots0\rangle\langle 0| + e^{i\alpha}|1\cdots1\rangle\langle 1|)\cancel{X}$ $\quad e^{i\alpha} X\otimes\cdots\otimes X(|0\cdots0\rangle\langle 0| + e^{-i\alpha}|1\cdots1\rangle\langle 1|)$

$=$ $\quad\quad =$

$|0\cdots0\rangle\langle 1| + |1\cdots1\rangle\langle 0|e^{i\alpha}$ $= e^{i\alpha}|1\cdots1\rangle\langle 0| + |0\cdots0\rangle\langle 1|$