

# Quantum Automated Theorem Proving

## Quantum automated theorem proving

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Automated theorem proving, or more broadly automated reasoning, aims at using computer programs to automatically prove or disprove mathematical theorems and logical statements. It takes on an essential role across a vast array of applications and the quest for enhanced theorem-proving capabilities remains a prominent pursuit in artificial intelligence. Here, we propose a generic framework for quantum automated theorem proving, where the intrinsic quantum superposition and entanglement features would lead to potential advantages. In particular, we introduce quantum representations of knowledge bases and propose corresponding reasoning algorithms for a variety of tasks. We show how automated reasoning can be achieved with quantum resolution in both propositional and first-order logic with quadratically reduced query complexity. In addition, we propose the quantum algebraic proving method for geometric theorems, extending Wu's algebraic approach beyond the classical setting. Through concrete examples, including geometry problems from the International Mathematical Olympiad, we demonstrate how a quantum computer may prove geometric theorems with **quadratic better query complexity**. Our results establish a primary approach towards building quantum automatic theorem provers, which would be crucial for practical applications of both near-term and future quantum technologies.

# Natives of the Pincus planet\*

## The problem

Travellers to the planet Pincus have observed that all its **healthy** natives are **noisy**, unless they are **fearful**. Those who are **fearful** and **quiet** are **happy**, as are those who are **healthy** and **noisy**. The **happy** and **quiet** natives are **healthy**, but those who are **fearful** and **healthy** are **unhappy**. Finally, even though the **unhappy** and **unhealthy** natives are always **afraid**, the **fearful** and **noisy** ones are **healthy**.



Fearful / Courageous



Happy / Unhappy



Unhealthy / Healthy



Noisy / Quiet

What can we deduce about the natives of the Pincus planet?

\*Problem adapted from *The Art of Computer Programming*, Volume 4, Donald E. Knuth

# Natives of the Pincus planet\*

The **conjunction** ( $x$  and  $y$ ):  $x \wedge y$  is true if  $x$  is true **and**  $y$  is true

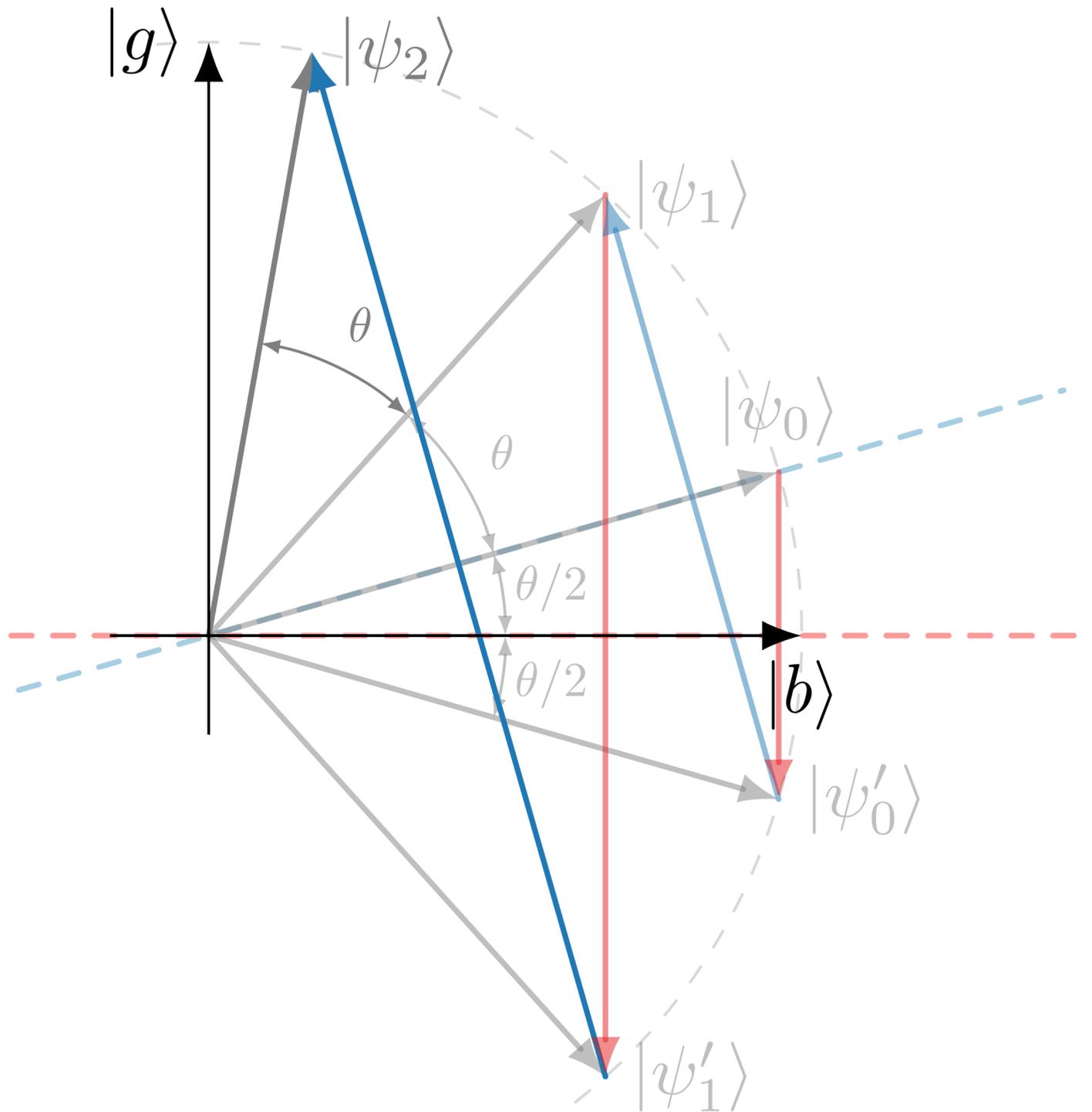
The **disjunction** ( $x$  or  $y$ ):  $x \vee y$  is true if  $x$  is true **or**  $y$  is true

The **negation** (**not**  $x$ ):  $\neg x$  is true if  $x$  is false (also written  $\bar{x}$  )

$$f(x_0, x_1, x_2, x_3) = (x_2 \vee x_0 \vee x_3) \wedge (\bar{x}_0 \vee x_3 \vee x_1) \wedge (x_2 \vee \bar{x}_3 \vee x_1) \wedge (\bar{x}_1 \vee x_3 \vee \bar{x}_2) \wedge (\bar{x}_0 \vee x_2 \vee \bar{x}_1) \wedge (x_1 \vee \bar{x}_2 \vee x_0) \wedge (\bar{x}_0 \vee \bar{x}_3 \vee \bar{x}_2)$$

		TRUE	FALSE
	$x_0$	Fearful	Courageous
	$x_1$	Happy	Unhappy
	$x_2$	Unhealthy	Healthy
	$x_3$	Noisy	Quiet

# Grover's algorithm



$$f(x_0, x_1, x_2, x_3) =$$

$$(x_2 \vee x_0 \vee x_3) \wedge (\bar{x}_0 \vee x_3 \vee x_1) \wedge$$

$$(x_2 \vee \bar{x}_3 \vee x_1) \wedge (\bar{x}_1 \vee x_3 \vee \bar{x}_2) \wedge$$

$$(\bar{x}_0 \vee x_2 \vee \bar{x}_1) \wedge (x_1 \vee \bar{x}_2 \vee x_0) \wedge$$

$$(\bar{x}_0 \vee \bar{x}_3 \vee \bar{x}_2)$$

Can find states which satisfy the logical statement  $f$  in  $O(\sqrt{N})$  queries

# Quantum Automated Theorem Proving

**a**

Do not play together (Alice's father and Bob's father)

Does not like (Alice and a badminton racket)

Play together (Alice and Bob)

Alice's father      Bob's father

Alice      Bob

**Knowledge base**

1. Anyone likes sports that their father likes.
2. If someone likes badminton, they likes a certain type of badminton racket.
3. Alice does not like any kind of badminton racket.
4. Badminton and basketball are sports.
- ...
9. Alice plays with her father
10. Bob plays with his father.

**Qestion:** Does Bob like both badminton and basketball?

**The badminton-basketball reasoning task**

**c**

**First-order logic representation**

$$S_1: \forall x, y \text{Likes}[\text{Father}(x), y] \wedge \text{Sport}(y) \rightarrow \text{Likes}(x, y)$$

$$\vdots$$

$$S_9: \text{Playtogether}[\text{Alice}, \text{Father}(\text{Alice})]$$

$$S_{10}: \text{Playtogether}[\text{Bob}, \text{Father}(\text{Bob})]$$

**Algebraic expressions**

$$h_1: x_4 x_6^2 + x_4 x_5 x_7 - x_4^2 x_6 = 0$$

$$\dots$$

$$h_2: x_6 x_8 + x_5 x_7 - x_4 x_6 = 0$$

$$\vdots$$

$$h_{16}: x_{22} + x_{20} - x_4 = 0$$

**Knowledge bases in formal languages**

**b**

**Knowledge base**

1. H is the orthocenter of the acute-angled triangle ABC.
2. D, E, F are the midpoints of BC, CA, AB, respectively.
3. The circle passing through H with center D intersects BC at points A<sub>1</sub> and A<sub>2</sub>.
4. B<sub>1</sub>, B<sub>2</sub>, C<sub>1</sub>, and C<sub>2</sub> are defined similarly.

**Task:** prove that A<sub>1</sub>, A<sub>2</sub>, B<sub>1</sub>, B<sub>2</sub>, C<sub>1</sub>, and C<sub>2</sub> lie on a circle.

**IMO 2008 Problem 1**

**Knowledge bases and reasoning tasks in natural languages**

**d**

**Encoding**

Running reasoning circuits

Updating knowledge base

Measuring and aquiring new knowledge

- Bob likes both badminton and basketball
- ...
- A<sub>1</sub>, A<sub>2</sub>, B<sub>1</sub>, B<sub>2</sub>, C<sub>1</sub>, and C<sub>2</sub> lie on a circle

# Conclusion

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- > quadratic quantum speedup
- > look inside
- > Grover's algorithm

